IOWA STATE UNIVERSITY
Digital Repository

1965

Scattering of electromagnetic waves from a known one-dimensional pressure field

Joallan Hootman
Iowa State University

www.manaraa.com

SCATTERING OF ELECTROMAGNETIC WAVES FROM A KNOWN

ONE-DIMENSIONAL PRESSURE FIELD

by

Joallan Hootman

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Electrical Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1965

TABLE OF CONTENTS

INTRODUCTION

Many times it is necessary to measure parameters of the atmosphere, such as temperature, pressure, and velocity, at locations where it is either impractical or dangerous to mount mechanical instruments and supports. For example, knowledge of the instantaneous wind profile along the flight path of an airplane as it approaches a landing strip would make the landing safer. Mounting mechanical instruments in the flight path to obtain this information would be a definite hazard. Also of interest are the atmospheric phenomena which take place in the scattering volume of a tropospheric communication link. Not only would the mounting of mechanical instruments in the volume be impractical, but the very presence of such instruments would distort the electromagnetic field and make accurate comparison of data difficult. Thus, a remote method of measuring atmospheric parameters in the scattering region is desirable.

The propagation of an acoustic wave is governed by various atmospheric parameters. Monitoring an acoustic wave as it propagates would give information about these atmospheric parameters. Because the monitoring is to be remote, a radar signal will be considered as the chief method of investigating the acoustic wave.

When an acoustic wave of finite length and of a known amplitude is launched into the atmosphere and illuminated with a radar signal, some back-scattering of the electromagnetic

energy will take place. The variation of the returned electromagnetic energy can give much information about the propagation characteristics of the acoustic waves under various atmospheric conditions. Atmospheric processes which affect the propagation and the amplitude of the acoustic wave thus can be measured. A method is needed which allows evaluation of the return signal of the radar from a finite length acoustic wave train propagating in a known atmosphere. From the information obtained by considering the acoustic wave in a well-defined atmosphere, a valid comparison can be made with a disturbed atmosphere, that is, an atmosphere where the pressure, temperature, and velocity are unknown.

Random pressure, temperature, and/or water vapor pressure (sic) disturbances, when illuminated by a radar signal, will reflect a part of the incident electromagnetic energy. In general, the details of the spatial and time distribution of the refractive index are unknown; therefore, calculation of the reflected electromagnetic energy is very difficult, if not impossible. If, however, the distribution of the refractive index is known and can be considered time as well as spatially stationary, then the reflection coefficient from this distribution can be found. This study considers the one-dimensional simplification of this problem.

The problem of electromagnetic energy reflection from a sinusoidal pressure wave will be considered first. The effect

of variation of the acoustic frequency deviations, magnitude
of the pressure fluctuations, and the length of the acoustic
pulse will be investigated and the influence of these factors
on the reflected electromagnetic energy noted.

High-powered acoustic pressure waves can be generated
easily in the atmosphere by the use of a siren or some form of
a whistle. The equations which govern the propagation of
these high-powered acoustic waves in the atmosphere are non-
linear. The nonlinear characteristics of the atmosphere cause
a progressive distortion of the acoustic wave as it moves away
from the generator. If it is necessary to use high-powered
acoustic waves to obtain a satisfactory reflected radar
signal, an investigation of the propagation characteristics
of this type of acoustic wave must be made. It will be seen
that for the case discussed in this thesis, the acoustic wave
must be considered high-powered. Mobile radar receivers and
transmitters are limited in power output and receiver
sensitivity; therefore, to obtain a predetermined amount of
returned signal, propagation of a high-powered acoustic wave
is desirable. The effect of the finite amplitude on the
acoustic wave will be discussed.

For most of the calculations in this thesis, the assump-
tion is made that the acoustic wave can be turned on and off
instantaneously. Physically, however, this assumption is not
entirely valid since the mechanical methods usually used to

generate the acoustic wave have finite build-up and decay times. Therefore, rather than being perfectly uniform, the acoustic wave will be modulated. The numerical technique to be presented here allows calculation of the reflection coefficient for any desired acoustic wave shape.

REVIEW OF LITERATURE

Two distinct problems are to be considered in this study. The first problem considers the reflection of electromagnetic energy from a known refractive index distribution. The second considers the propagation of a finite amplitude acoustic wave. A detailed numerical study of the finite amplitude acoustic wave was not attempted. Rather, an analytical method of attack on a finite amplitude acoustic wave is presented and the necessary information about the pressure distribution obtained from this approach.

Brillouin (3) presents a detailed discussion of the propagation of waves in periodically stratified media. The solution of Mathieu's and Hill's equation is duscussed. The classical treatments of these two equations assume that the periodic media are infinite in extent. Solution of the Mathieu equation for a finite length of the periodic media can be accomplished; but, in general, for the accuracy required and the numbers involved, a great amount of computer time is necessary. A faster method of numerical solution of this specific problem is needed.

Barrar and Redheffer (1) made a study of the fields in a dielectric medium which varied in one dimension. Their approach was to consider the media to be sectioned into parallel homogeneous slabs. The fields inside the slabs were then found. From these fields a differential equation

describing the wave impedance was found. This equation was then solved to give the reflection and transmission coefficients of a system. Schmitt and Wu (12, 13) considered the case of an electromagnetic wave incident upon a tank filled with oil into which an acoustic wave had been launched. The formulation presented was the solution of the standard Mathieu equation; verification of the formulation was obtained experimentally. Jones and Patton (8) considered a transmission line analogy to a sinusoidal variation of the relative dielectric constant. Their system involved the solution of the Riccati equation. Jones and Patton gave data for the magnitude of the reflection coefficient.

Richmond (11) considered slabs of dielectric and derived a set of difference equations for a numerical solution of the wave equation in the slab. Reflection and transmission coefficients were found and an error analysis performed to find the best step size for use in the computer solution of the difference equations. Fetter, Smith, and Klein (6) conducted both experimental and analytical work on the problem of tracking an acoustic burst with a doppler radar system. Their study was an attempt at remote measurement of the wind velocity of the atmosphere.

Relatively large variations of the dielectric constant have been considered by most investigators. However, Jones and Patton (8) considered the finite length of the acoustic wave as well as small magnitude changes in the refractive index.

Fat (5) made a study of high-powered acoustic waves by
an intricate method of Fourier series analysis. He concluded
that, as the wave propagates away from the source, an
initially sinusoidal wave gradually distorts into a sawtooth-
shaped wave. More recently Blackstock (2) has also shown that
if an intensity of 160-170 db (reference intensity of 10^{-16}
watts/cm^2) is used, a sawtooth wave does occur in the limit
of large distances. For high intensity acoustic waves, above
100 db, the wave shape distorts into a sawtooth wave. As
the power is increased, the distortion will occur closer to
the source.

MATHEMATICAL DEVELOPMENT OF THE PROBLEM

Development of the Analytical Description of the

Electromagnetic Acoustic System

The method of development of the necessary equations to describe the interaction of the electromagnetic wave with a one-dimensional distribution of the refractive index follows, in large measure, the development presented in Richmond's paper (11).

Maxwell's equations will be used, with the following conditions assumed to hold:

1. there is no free charge in the atmosphere.

2. the permeability, μ, is spatially constant, independent of the acoustic wave.

3. the atmosphere is a lossless dielectric medium.

4. the refractive index disturbance is assumed to be both time and spatially stationary in relation to the electromagnetic wave.

5. the electromagnetic wave propagates as a uniform plane wave with its direction of propagation in the x-direction.

6. the permittivity, ϵ, of the atmosphere varies only with x.

The assumed restrictions on the media and the electromagnetic wave allow simplification of the equations which will be derived. Essentially these restrictions on the media require that the disturbance, at the instant it is illuminated

by the electromagnetic source, be momentarily fixed in time

and space. In keeping with these assumptions, the permittivity

is assumed to have no time dependence.

Maxwell's equations, with the above assumptions taken

into account, thus can be stated as:

$$\nabla \times \vec{\mathcal{E}} = - \frac{\partial \vec{\mathcal{B}}}{\partial t} = - \mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \tag{1}$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t} \cong \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{\mathcal{D}} = 0 \tag{3}$$

$$\nabla \cdot \vec{\mathcal{B}} = 0 \tag{4}$$

The curl of Equation 1, combined with Equation 2 and

the vector identity $\nabla \times (\nabla \times \vec{\mathcal{H}})$, gives:

$$\nabla \times (\nabla \times \vec{\mathcal{E}}) = \nabla (\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}} = -\mu \frac{\partial (\nabla \times \vec{\mathcal{H}})}{\partial t} = -\mu \epsilon \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} \tag{5}$$

Equation 3 can be written as:

$$\nabla \cdot \epsilon \vec{\mathcal{E}} = \vec{\mathcal{E}} \cdot \nabla \epsilon + \epsilon \nabla \cdot \vec{\mathcal{E}} = 0 \tag{6}$$

From Assumption 5, the electric field will be of the form

$$\vec{\mathcal{E}} = \text{Re} \left[\sqrt{2} \left\{ \bar{U}y \, \bar{E} \exp (\pm j\omega t) \right] \right. \tag{7}$$

where $\bar{E}$ is a phosor. From Assumption 6 $\vec{\mathcal{E}} \cdot \nabla \epsilon = 0$ therefore

$\nabla \cdot \bar{E} = 0$. Making these substitutions, Equation 5 becomes

$$\frac{d^2 \bar{E}}{dx^2} = -\mu \epsilon \omega^2 \bar{E} = -k_o^2 \, \epsilon_r \bar{E} \tag{8}$$

where $k_o = \frac{\omega}{c_o} = \frac{2\pi}{\lambda_o}$, $c_o = \frac{1}{\sqrt{\mu_o \epsilon_o}}$, ω is the angular frequency

of the electromagnetic wave, λ_o is the electromagnetic wave length, and k_o is the electromagnetic wave number. The relative dielectric constant ϵ_r is the x-dependent variable in an otherwise classical one-dimensional wave equation. Equation 8 must hold inside the region of interest as well as satisfy the boundary conditions.

In the atmosphere the relative permittivity can be expressed as a function of pressure (P), temperature (T), and water vapor pressure (W). Smith and Weintraub (14) give the relative permittivity as $\epsilon_r = n^2$, where $n = 1 + f(P,T,W)(10^{-6})$ $= 1 + f(10^{-6})$. The term $f(10^{-6})$ is normally a small quantity. If typical sea level conditions are assumed, $f(10^{-6}) \simeq 3(10^{-4})$; hence, $\epsilon_r \simeq 1 + 2 f(10^{-6})$, when second-order terms are neglected.

The interaction of $f(10^{-6})$ with the incident electromagnetic field will, in general, be small; thus, a perturbation form of the solution of the electric field, Equation 8, can be made. The assumption, therefore, is made that:

$$\bar{E} = \bar{E}_o + \bar{E}_1 \tag{9}$$

where $\bar{E}$ is the total field and $\bar{E}_1$ is the perturbation field caused by the interaction of f and $\bar{E}_o$. $\bar{E}_1$ will be much smaller than $\bar{E}_o$, and it will be assumed that terms of the order $(\bar{E}_1)(f)(10^{-6})$ will be negligible.

Direct substitution of Equation 9 into 8 and the neglecting of second-order terms reduce Equation 8 to:

$$\frac{d^2\bar{E}_o}{dx^2} + \frac{d^2\bar{E}_1}{dx^2} = -k_o^2 \ [\bar{E}_o + 2f(10^{-6})\bar{E}_o + \bar{E}_1] \tag{10}$$

Since $\bar{E}_o$ is the solution to the wave equation which must hold in a region where no spatial variation of $\bar{E}_1$ exists, Equation 10 can be separated into two equations, _viz_,

$$\frac{d^2\bar{E}_1}{dx^2} + k_o^2\bar{E}_1 = -k_o^2\bar{E}_o \ 2f(10^{-6}) \tag{11}$$

$$\frac{d^2\bar{E}_o}{dx^2} = -k_o^2\bar{E}_o \tag{12}$$

If the solution for Equation 12 is a traveling wave of the form $\bar{E}_o e^{jk_o x}$ and Equation 11 is normalized by dividing through by $\bar{E}_o$, $\bar{E}_1$ can be expressed in terms of its normalized real and imaginary parts $\frac{\bar{E}_1}{\bar{E}_o} = R+jI$. Equations 13 and 14 follow when it is assumed that the function f is real:

$$\frac{d^2R}{dx^2} + k_o^2 R = 2f(10^{-6}) \ k_o^2 \cos k_o x \tag{13}$$

$$\frac{d^2I}{dx^2} + k_o^2 I = 2f(10^{-6}) \ k_o^2 \sin k_o x \tag{14}$$

If $f(10^{-6})$ is known, Equations 11 and 12 can be solved readily, and the total filed inside the region of interest can be found from Equation 9. Only a solution for Equations 13 or 14 is needed as the equations are in phase quadrature with respect to time. The discussion to follow, however, assumes both of the equations are to be solved.

The reflection coefficient can be found once $\bar{E}_1$ is known.

Reflection Coefficient in Terms of the Perturbed Field $\bar{E}_1$

The scattering region, that is, the region where the refractive index varies, is shown in Figure 1 where $\bar{E}_i$, $\bar{E}_r$, and $\bar{E}_t$ are the phasor notations for the incident, reflected, and transmitted electric fields respectively.

At boundary A, $\bar{E}_i|_{x=A} + \bar{E}_r|_{x=A} = \bar{E}_o|_{x=A} + \bar{E}_1|_{x=A}$ (15)

At boundary B, $\bar{E}_t|_{x=B} = \bar{E}_o|_{x=B} = 1 <o$ (16)

It is assumed here that $\bar{E}_1$ is zero until f is of sufficient magnitude to interact with $\bar{E}_o$ to produce an $\bar{E}_1$. This assumption obviously requires that $\bar{E}_o = \bar{E}_t$ at boundary B.

The transmitted wave $\bar{E}_t$ may be assumed to be any value consistent with the boundary condition B and the normalization used in Equations 13 and 14. The transmitted field is assumed, for convenience, to have a magnitude of unity and an associated phase angle of zero.

The standard equations for the transmission and reflection coefficients are expressed by Ramo and Whinnery (10) as:
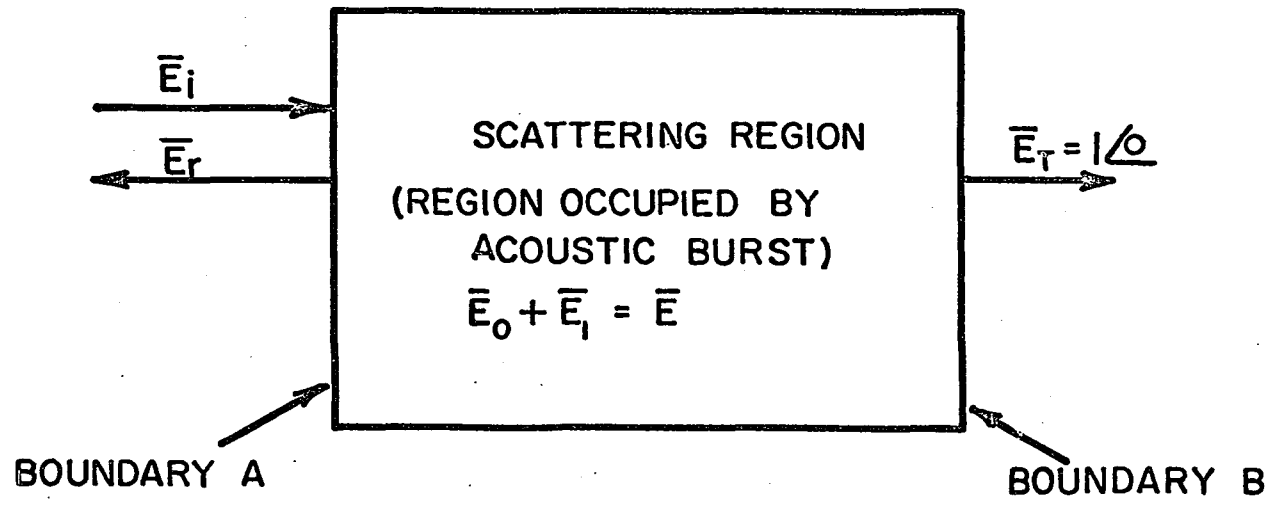
$$\bar{\rho} = \frac{\bar{E}_r}{\bar{E}_i} \tag{17}$$

$$\bar{T} = \bar{E}_t / \bar{E}_i \tag{18}$$

$$\bar{T} = 1 - \bar{\rho} \tag{19}$$

These equations along with the boundary conditions will be used to find the reflection coefficient. Elimination of $\bar{T}$ and $\bar{\rho}$ from Equations 17, 18, and 19 gives the obvious result that, at any point x:

Figure 1.   Schematic representation of the scattering region used
             to find the expression for the reflection coefficient
             as a function of the perturbed field.

$$\bar{E}_i$$

$$\bar{E}_r$$

SCATTERING REGION

(REGION OCCUPIED BY
ACOUSTIC BURST)

$$\bar{E}_o + \bar{E}_i = \bar{E}$$

$$\bar{E}_T = 1 \angle 0$$

BOUNDARY A

BOUNDARY B

$$\bar{E}_i - \bar{E}_r = 1 \tag{20}$$

Substituting boundary condition 16 into 15 gives:

$$\bar{E}_i + \bar{E}_r = 1 + \bar{E}_1 \tag{21}$$

When Equation 21 is subtracted from Equation 20, an expression for $\bar{E}_r$ is obtained. The reflection and transmission coefficients can be written as:

$$\bar{\rho} = \bar{E}_1/2\bar{E}_i \simeq \bar{E}_1/2 \tag{22}$$

$$\bar{T} \simeq 1 - \bar{E}_1/2 \tag{23}$$

when $\bar{E}_i \simeq 1$. Later calculations will show that, for the problems of practical interest, the reflected power is always much smaller than the incident and transmitted power; hence, $\bar{E}_i \simeq 1$, which justifies the approximation used to obtain Equations 22 and 23. As previously stated, it is the reflection coefficient, Equation 22, that is of interest.

The Method of Numerical Solution of the Differential Equation

The perturbed wave Equations 13 and 14, although linear, have spatially periodic coefficients and, hence, are not conveniently solved by a simple technique. However, by the use of a standard program for numerical solution of such equations, a solution can be implemented easily.

The standard library Runge-Kutta program used to solve these equations requires that the second-order differential equation be separated into two first-order differential

equations. In Equation 13, substituting $Z_1 = \frac{dR}{dx}$ and $Z_2 = R$ results in:

$$\frac{dZ_1}{dx} + k_o^2 Z_2 = -2f(10^{-6}) \, k_o^2 \cos k_o x \tag{24}$$

$$Z_1 = \frac{dZ_2}{dx} \tag{25}$$

Equations 24 and 25 are thus coupled first-order differential equations which must be satisfied in the region of interest and which must be solved together. This same type of substitution can be used on Equation 14, with the resulting expression:

$$\frac{dZ_3}{dx} + k_o^2 \, Z_4 = -2f(10^{-6}) \, k_o^2 \sin k_o x \tag{26}$$

$$Z_3 = \frac{dZ_4}{dx} \tag{27}$$

To obtain a unique solution to a second-order differential system, two boundary conditions must be established. Therefore, to complete the solution of the perturbed second-order differential equation, two boundary conditions must be found. These boundary conditions will be used in conjunction with the respective pairs of equations, Equations 24 and 25 and Equations 26 and 27.

Initial conditions

For the numerical program used, it is necessary to find values of Z_1, Z_2, Z_3, and Z_4 at the point $x = 0$, hereafter described as the origin. For convenience the origin is chosen as a point on the leading edge of the acoustic burst as it

propagates away from the transmitter. To find these values, a Taylor series expansion of $\bar{E}$ and $\bar{E}_0$ will be made at a very small distance, h, from the origin into the region of scattering (see Figure 2). In the Taylor series expansion, the coefficients of the expansion are evaluated at x = 0, and the expansion is evaluated at x = h.

If E is a wave of the form e^{jk_0x}, the Taylor expansion for $\bar{E}(h)$ becomes

$$\bar{E}(h) = 1 - \frac{h^2}{2} \epsilon_e k_0^2 - \frac{h^3}{6} k_0^2 \epsilon_r' + j [hk_0 - \frac{h^3}{6} \epsilon_r k_0^3]$$

where ϵ_r' is the spatial derivative of the dielectric constant. When $\bar{E}_0$ is expanded in a similar manner,

$$1 - (h^2/2) k_0^2 + j [k_0 h - (h^3/6) k_0^3] .$$

From the assumed Equation 9 for the total field, $\bar{E}$, and the approximate equation for ϵ_r, $\bar{E}_1$ is found to be:

$$\bar{E}_1(h) = -[h^2 k_0^2 f|_{x=h} + \frac{k_0^2 h^3}{3} f'|_{x=h}](10^{-6}) + j \frac{h^3 k_0^3}{3} f|_{x=h}(10^{-6})$$

$$(28)$$

The ratio of the imaginary part of the series expansion to the real part of the expansion is approximately $hk_0 = 2\pi h/\lambda_0$. For small values of h/λ_0, the imaginary part of the series expansion of $\bar{E}_1(h)$ can be neglected.
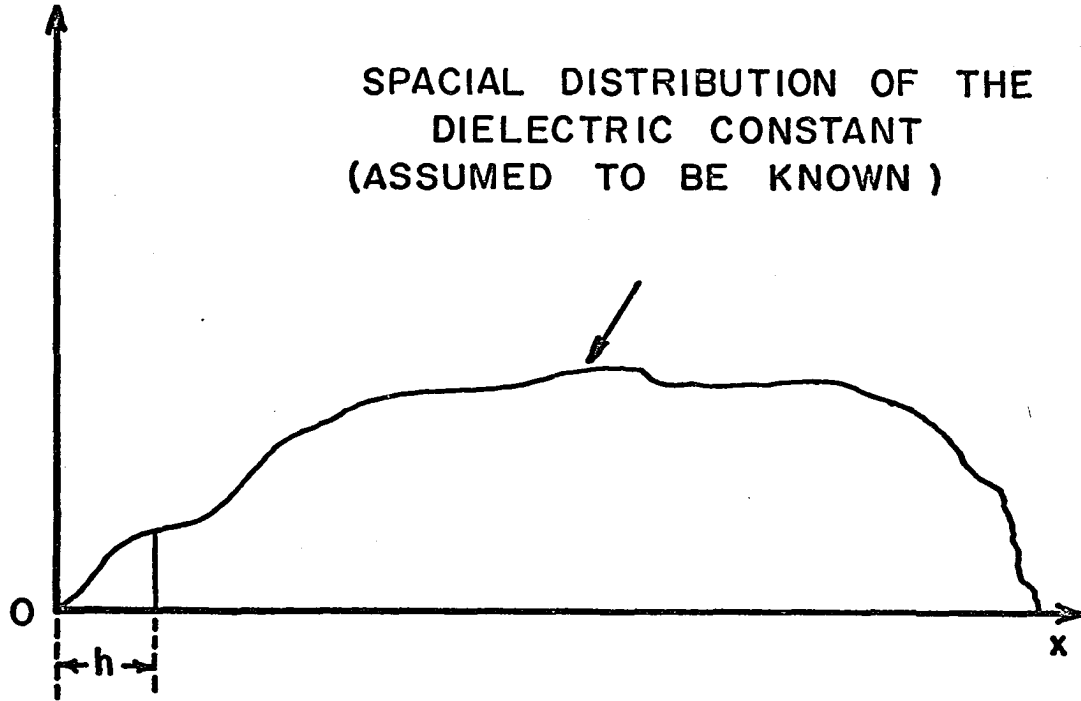
The perturbed field $\bar{E}_1(x)$ in the small distance, h, from the origin is assumed to have the form of

$$\bar{E}_1(x) = R + jI = [ReE_1(h)](\cos k_0 x + j \sin k_0 x) \text{ for } (0 \leq x \leq h),$$

Figure 2. Schematic representation of a known distribution of the
dielectric constant used in the derivation of the
boundary conditions.

$\epsilon_r - 1$, $2 fx(10^{-6})$

SPACIAL DISTRIBUTION OF THE
DIELECTRIC CONSTANT
(ASSUMED TO BE KNOWN)

O

|←h→|

x

where $\mathrm{Re}\bar{E}_1(h)$ is the real part of Equation 28, the series expansion of $\bar{E}_1(h)$.

At the origin, the spatial derivative of the real part of $\bar{E}_1(x)$ will be zero; therefore, $Z_1\big|_{x=0}=0$. However, there will be a real component of the $\bar{E}_1(x)$ field at $x=0$ so that $Z_2\big|_{x=0}=\mathrm{Re}[\bar{E}_1(h)]$.

At the origin, the imaginary part of E_1 will be zero; thus, $Z_4\big|_{x=0}=0$. There will be a value of the spatial derivative of $\bar{E}_1(x)$ so that $Z_3\big|_{x=0}=k_o\mathrm{Re}[E_1(h)]$.

In summary, the boundary conditions are:

For the real part of $\bar{E}_1(x)$ at $x=0$

$$Z_2 \cong -(h^2k_o^2f\big|_{x=h}+\frac{k_o^2h^3}{3}f'\big|_{x=h})(10^{-6})$$

$$\frac{dZ_2}{dx} = Z_1\big|_{x=0}=0 \tag{29}$$

For the imaginary part of $\bar{E}_1(x)$ at $x=0$

$$\frac{dZ_4}{dx} \cong -k_o(h^2k_o^2f\big|_{x=h} + \frac{h^3k_o^2}{3}f'\big|_{x=h})(10^{-6}) \tag{30}$$

$$Z_4 = 0$$

The value of h/λ_o must be small to give a good estimate of the boundary conditions. The solution for the reflection coefficient, however, is relatively insensitive to the boundary conditions if they are of the correct order of magnitude.

Thus far, only limited mathematical restrictions have been placed on the function f: f must be a known one-dimensional function which is continuous and possesses continuous

derivatives in a region h. In the derivation of the boundary

conditions, it is tacitly assumed that the higher-order

derivatives can be neglected. The physical phenomena which

produce the fluctuation in $f(10^{-6})$ have not been limited in

any sense. However, to check the formulation, the effect of

initial conditions, and the reflection from a finite length

periodic pressure wave, the assumption will be made that

$2f(10^{-6})$ in Equations 24 through 27 is of the form $10^{-6} \sin k_a x$,

where $k_a = \dfrac{2\pi}{\lambda_a} = \dfrac{\omega}{c_a}$ is the acoustic wave number.

Approximation of the Reflection Coefficient from a Sinusoidal Pressure Field

A sinusoidal distribution of pressure will now be assumed

in order to develop an analytical technique which allows a

determination of the sensitivity of the reflection coefficient

to parameters which can be varied experimentally. The form of

the pressure distribution, in keeping with the previous

assumptions, will be $K \sin k_a x$, where K is a constant.

Therefore, $n = 1 + 10^{-6} \sin k_a x$.

For the derivation of the reflection coefficient, the

model shown in Figure 3 will be used, and the equation for the

reflection coefficient from a lossless dielectric will be

assumed to apply:

$$\bar{\rho} = \frac{Z_1 - Z_o}{Z_1 + Z_o} = \frac{1-n}{1+n} \tag{31}$$

Figure 3. Schematic representation of the refractive
           index used to derive an analytical approximation
           for the reflection coefficient.
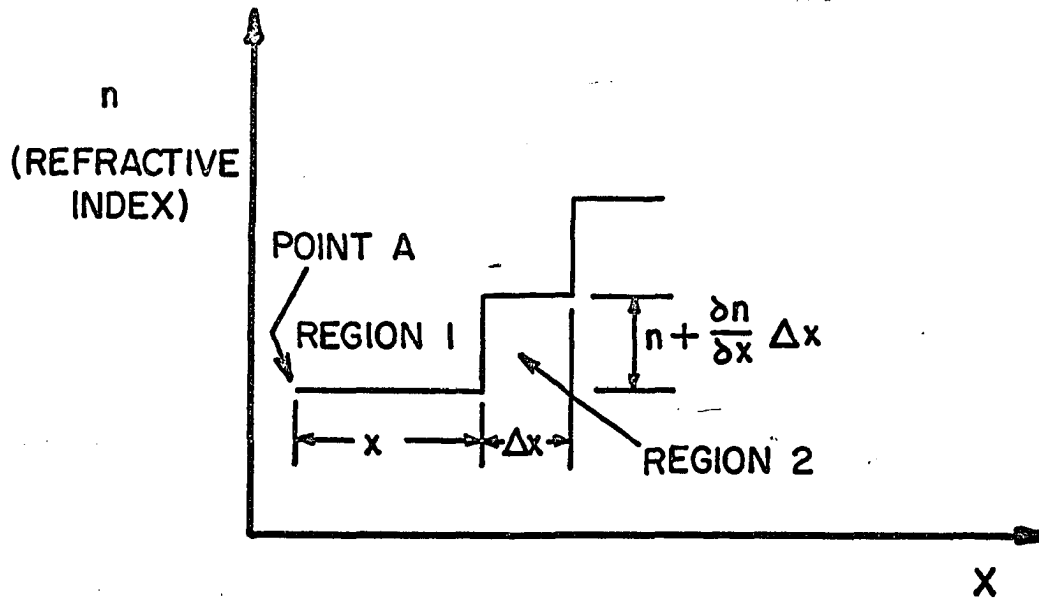
The approach to be followed in the development is to find, in general, a $\Delta\bar{\rho}$ and its associated phase angle by assuming single reflections and using Equation 31. A $\bar{\rho}$, which is an approximate reflection coefficient for the acoustic burst, is found by taking the limit of the expression for the $\Delta\bar{\rho}$'s, and integrating.

A wave which is measured at point A (see Figure 3) will travel a distance 2x to and from the disturbance and, thus, will contribute a phase factor to the reflection coefficient of $e^{\pm j2k_0 x}$. The change in $\bar{\rho}$ can be written as follows:

$$\Delta\bar{\rho} = \frac{1 - (n + \frac{\partial n}{\partial x}\Delta x)}{1 + (n + \frac{\partial n}{\partial x}\Delta x)}\ e^{\pm jk_0 x} \tag{32}$$

In the limit, the refractive index, n, will change very little from unity. Thus,

$$|\bar{\rho}| = \tfrac{1}{2}\left|\int_0^L \frac{\partial n}{\partial x}\ e^{\pm j2k_0 x}dx\right| \tag{33}$$

where L is the overall length of the perturbed region of n or the acoustic pulse length. Equation 33 is similar to that obtained by Smith and Rogers (15). If N is the number of acoustic cycles and $\bar{\lambda}_a$ is the acoustic wave length, L = $N\bar{\lambda}_a = \frac{N2\pi}{k_a}$. Because n is assumed to be sinusoidal, Equation 33 will have a principal maxima when $k_a = 2k_0$:

$$|\bar{\rho}| = \left|\frac{Kk_a}{2}\int_0^L \cos^2 k_a x\ dx\right| = \left|\frac{KN\pi}{2}\right| \tag{34}$$

For a sinusoidal pressure distribution, the reflection coefficient will increase linearly with the magnitude of the

pressure and the number of acoustic cycles, that is, the length of the acoustic pulse.

Small Angles of Incidence and Acoustic Frequency Change

A small change in the direction of propagation of a sinusoidal acoustic pressure wave, being tracked by radar, will mean that the electromagnetic energy will make a small angle of incidence with respect to the direction of propagation of the acoustic wave. One of the effects of the change in the angle of incidence is to make an apparent change in the acoustic wave length.

To begin a study of the effect of the angle of incidence on the reflection coefficient, the following approximations are made:

1. the angle of incidence is small.
2. the angle of the radar reflection is such that it can be picked up by the receiver.
3. the transmitted angle is the same as the incident angle.

Approximation 2 implies Approximation 1, but this assumption, nevertheless, must be kept in mind. At some large distance between the transmitter and receiver, the small angle assumption still may be valid; but it is possible that no reflected signal would be incident on the receiver antenna. Approximation 3 can be shown to hold for the case of interest by using Snell's Law, which states: $\dfrac{\sin \theta_1}{\sin \theta_2} = [\epsilon_2/\epsilon_1]^{\frac{1}{2}} = n_2/n_1$ where θ_1, θ_2, ϵ_1
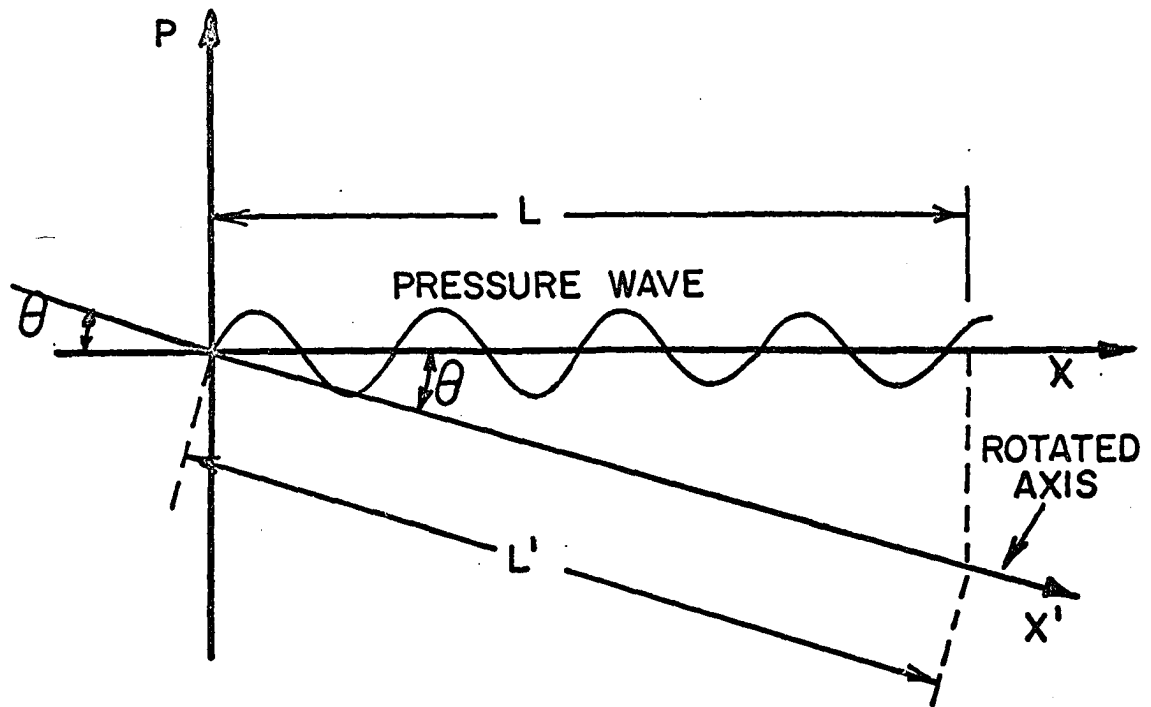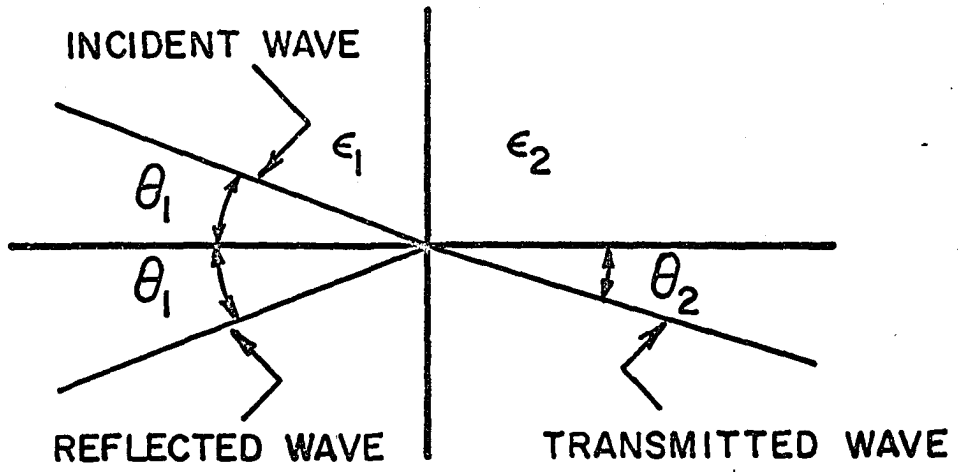
and ϵ_2 are defined in Figure 4. For the extreme case which might be encountered, $n_1 = 1 + K(10^{-6})$ and $n_2 = 1 - K(10^{-6})$. For the order of magnitude of the K's considered, $n_1 \simeq n_2$; therefore, $\theta_1 \simeq \theta_2$, and Approximation 3 is then justified.

Because the transmitted angle and the incident angle can be considered equal, an axis rotation can be made. If a rotation of the axis is performed as shown in Figure 5, then to the electromagnetic wave propagating along the x' axis, the apparent wave length of the acoustic wave increases. The acoustic wave number k_a', for the rotated coordinate axis, is $k_a' = k_a \cos \theta$. The length, $L' = L/\cos \theta$, is the interaction length of the electromagnetic wave with the acoustic wave. For small angles, the reflection coefficient should not change radically.

If the acoustic frequency were to change and the velocity of acoustic propagation remain constant, k_a would change. The change in the acoustic wave number caused by the angle of incidence could be construed as an apparent change in the acoustic frequency. For large values of $\cos \theta$, Approximations 1 and 2 do not hold; but the numerical data obtained can be used to find the effect of a shift in the acoustic frequency. Both the acoustic and electromagnetic waves are assumed to be plane waves for purposes of calculating the variation of the reflection coefficient with acoustic frequency.

Figure 4. Schematic diagram used to indicate quantities used in Snell's Law.

Figure 5. Schematic diagram of the axis rotation used for small angles of incidence.

INCIDENT WAVE

ϵ_1

ϵ_2

θ_1

θ_1

θ_2

REFLECTED WAVE

TRANSMITTED WAVE

P

L

PRESSURE WAVE

θ

θ

X

ROTATED AXIS

L'

X'

Reflection from Modulated Acoustic Waves

In the discussion to this point, the assumption was made that the acoustic wave could be instantaneously started and stopped; in most cases, this assumption is not valid. Mechanical methods which are used to generate the acoustic wave usually have finite rise and decay times associated with them. The problem of finding the reflection of the electric field from a modulated sinusoidal wave is handled easily with the previously presented formulation.

Three cases of the modulated acoustic burst will be used as examples. The values of the reflection coefficients and the graphs of the perturbed field will be given in the section on Calculations and Data.

The first case considered is that of an acoustic wave which is half sine wave modulated as shown in Figure 6. In Equations 24 and 25, the expression $2f(10^{-6}) = 10^{-6} \sin k_a x \sin \frac{\pi}{L} x$ is substituted.
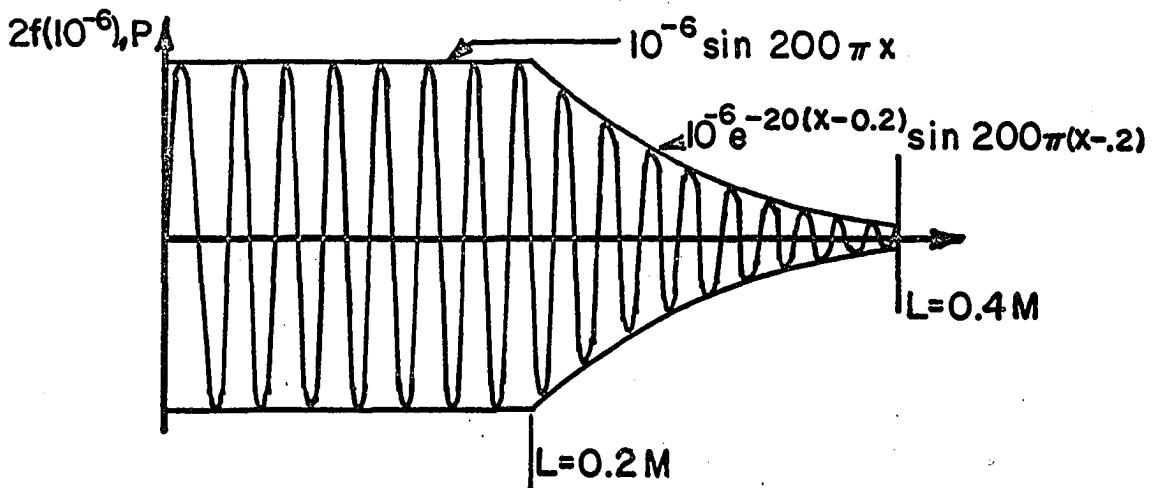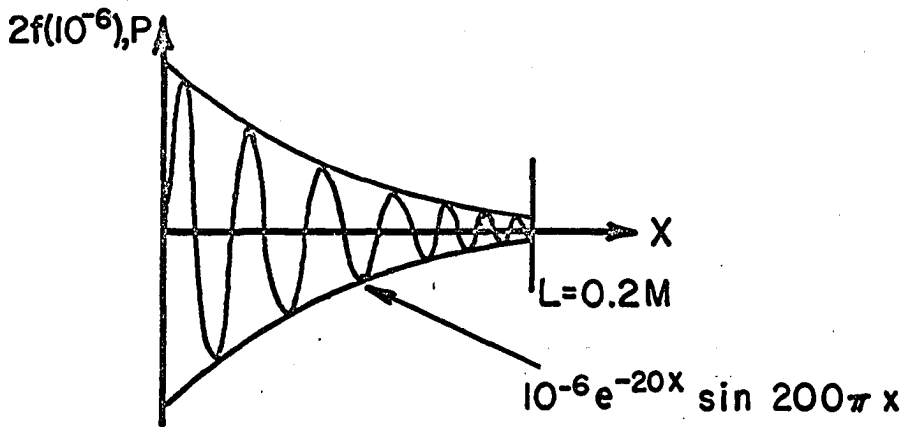
The second case considered is a damped sinusoidal wave starting at the origin. In this instance, the expression $2f(10^{-6}) = 10^{-6} e^{-\alpha x} \sin k_a x$ is substituted into Equations 24 and 25. This wave is shown in Figure 7. The α is chosen to be compatible with the acoustic decay which might be expected from a mechanical source.

The final case to be studied is an acoustic wave which is not modulated until some distance L; then the acoustic wave

Figure 6. Schematic representation of an acoustic burst modulated by a one-half sine wave.

Figure 7. Schematic representation of an acoustic burst modulated by a damped exponential.

Figure 8. Schematic representation of a practical modulated acoustic wave form.

$2f(10^{-6}),P$ X   L = 0.2 M $10^{-6}\sin 5\pi x \sin 200\pi x$

$2f(10^{-6}),P$ X   L = 0.2M $10^{-6}e^{-20x}\sin 200\pi x$

$2f(10^{-6}),P$ $10^{-6}\sin 200\pi x$ $10^{-6}e^{-20(x-0.2)}\sin 200\pi(x-.2)$ L=0.4M   L=0.2M
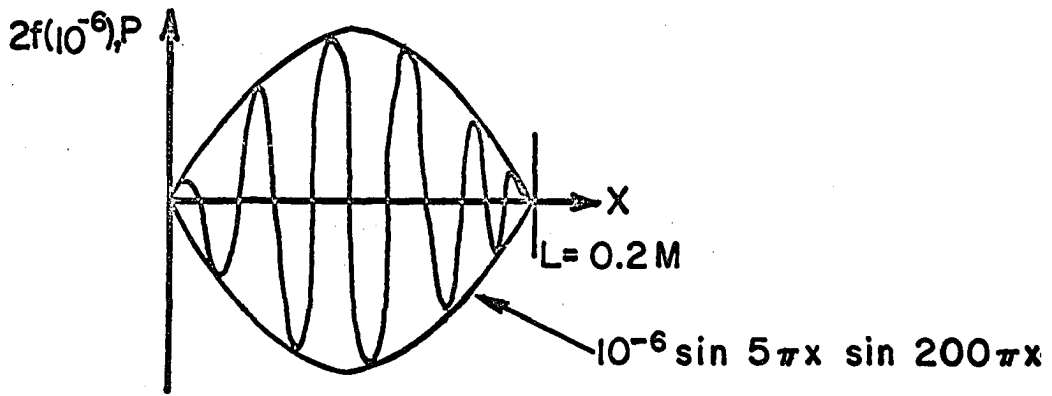
is allowed to decay in an exponential manner. The α is the

same α used in the second case. This wave shape is close

to what would be expected in a practical situation. This wave

form is shown in Figure 8.

CALCULATIONS AND DATA

Three values of the reflection coefficient were cal-
culated and presented by Jones and Patton (8). Their values
were $7.64(10^{-6})$, $7.86(10^{-6})$, and $7.91(10^{-6})$. Their first
value was obtained by considering the general problem of
reflection from a sinusoidally stratified dielectric in terms
of a transmission line analogy. Formulation and solution of
the problem in terms of the Mathieu function gave results of
$7.91(10^{-6})$ for the reflection coefficient. An analytical
method (sic), which they did not describe, yielded 7.86
(10^{-6}) as the reflection coefficient. These values of the
reflection coefficient will be compared with the values
obtained by the use of the technique presented here. In
keeping with the normalization used by Jones and Patton,
$k_a = 200\pi$, $k_o = 100\pi$, and $2f(10^{-6}) = 10^{-6} \sin k_o x$ were
substituted in Equations 24 through 27; and x was allowed to
vary from zero to 0.2 meters. In the evaluation of the
initial conditions, hk_o in Equations 29 and 30 was assumed to
be 0.174 radians, that is, approximately 10 degrees.

From such information, a complete solution for this
specific problem can be found. A subroutine which allows
numerical evaluation of the equations is easily written.
The reflection coefficient obtained from the formulation
presented here was $|\rho| = 7.86(10^{-6})$, which is well within
the bounds of Jones and Patton's published results. A graph

of the real and imaginary parts of this solution over half the region of interaction, from the origin to 0.1 meters, is shown in Figure 9.

The effect of the initial conditions is most noticeable near the origin. If a solution near the origin is needed (as, for example, in the case of a short acoustic burst), more accurate boundary conditions are needed. Using smaller values of h to evaluate the boundary conditions will give a more accurate solution at the origin. More computer time, however, might be required to provide the needed accuracy in this region.

The sensitivity of the solution to the initial conditions was checked by dividing and multiplying the initial conditions by a factor of 5. The results are shown in Table 1. Inspection of Table 1 shows that the boundary conditions have little effect

Table 1. Effects of boundary conditions on $Re[E_1]$ at $x=2(10^{-2})$ meters

Boundary Conditions	$Re[E_1]$ at $x = 2(10^{-2})$ meters
$Re[E_1(h)]$	1.570×10^{-6}
$Re[E_1(h)]/5$	1.5707×10^{-6}
$Re[E_1(h)](5)$	1.5686×10^{-6}

on the value of the solution at a distance as short as one-half of an acoustic wavelength. As the point of evaluation moves away from the origin, the effects of the boundary

Figure 9.   A plot of the normalized perturbed scattering field as a
            function of distance for a sinusoidally varying pressure wave.
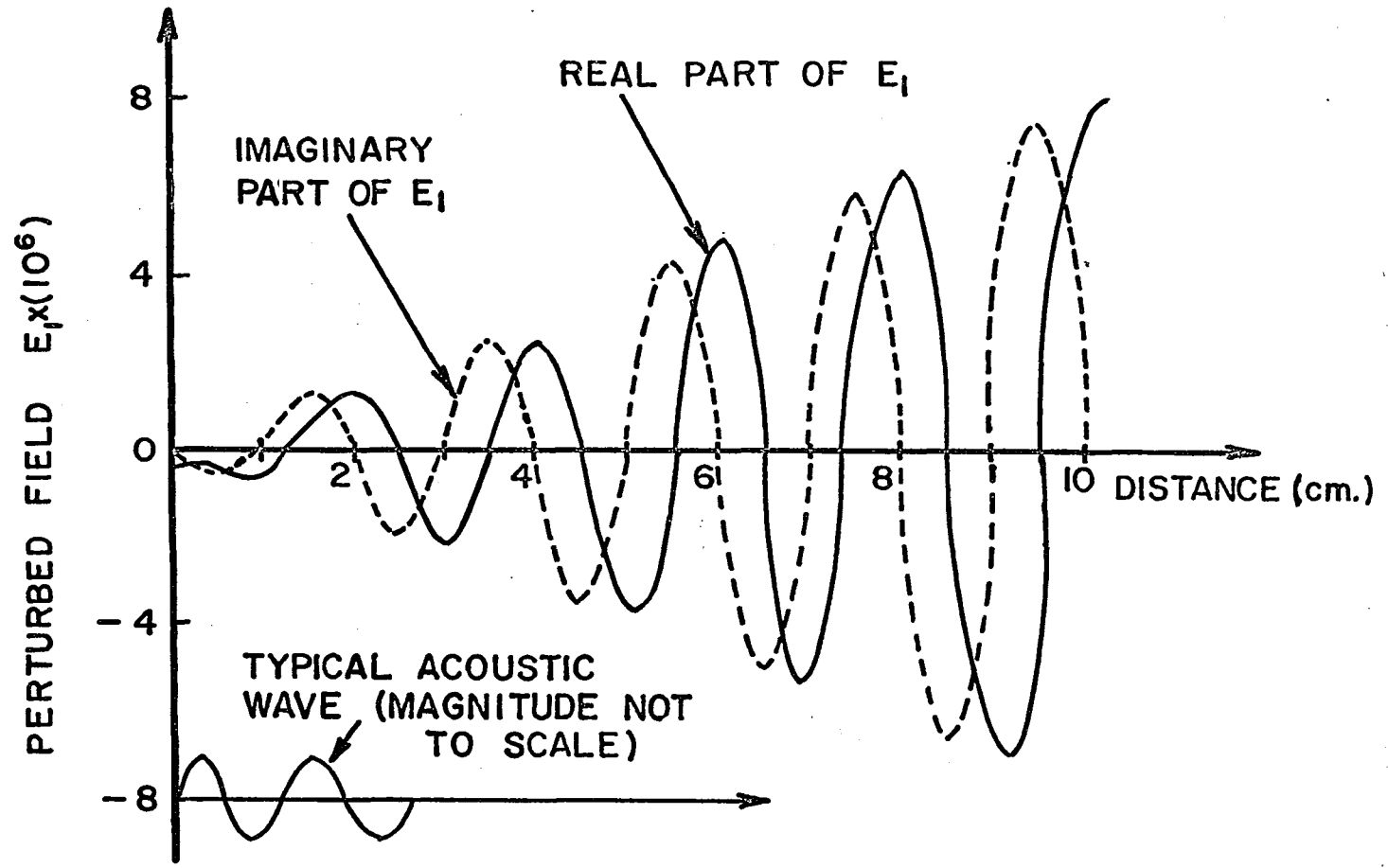
conditions tend to be even less evident.

Equation 34 predicts that the magnitude of the reflection coefficient varies in a linear manner with the magnitude of the acoustic pressure and the number of acoustic wavelengths that are illuminated by the electromagnetic energy. As a check on the sensitivity of the reflection coefficient to the parameters of the acoustic burst, as predicted by Equation 34, $2f(10^{-6})$ was set equal to $\alpha 10^{-6} \sin k_a x$. The graph of the variation of the reflection coefficient for different values of α is given as Figure 10. Variation of the magnitude of the reflection coefficient with the number of cycles of the acoustic burst is the same as the variation of the magnitude of the reflection coefficient with the distance in the region of interaction. A plot of the reflection coefficient magnitude as a function of distance for $\alpha = 1$ is shown in Figure 11. The functional agreement with Equation 34 is evident.

As the frequency of the acoustic wave changes, the reflection coefficient should change. When $k_a = 2k_o$, the reflection of electromagnetic energy is maximum. If this relationship changes, the reflection of electromagnetic energy will change; thus, the reflection coefficient will change. It has been shown previously that the data obtained from a change in the angle of incidence can be construed as a change in the acoustic frequency.

Figure 10.   The graph of the magnitude of the
             reflection coefficient to the variation
             of the refractive index, $n = 1 + \alpha 10^{-6}$
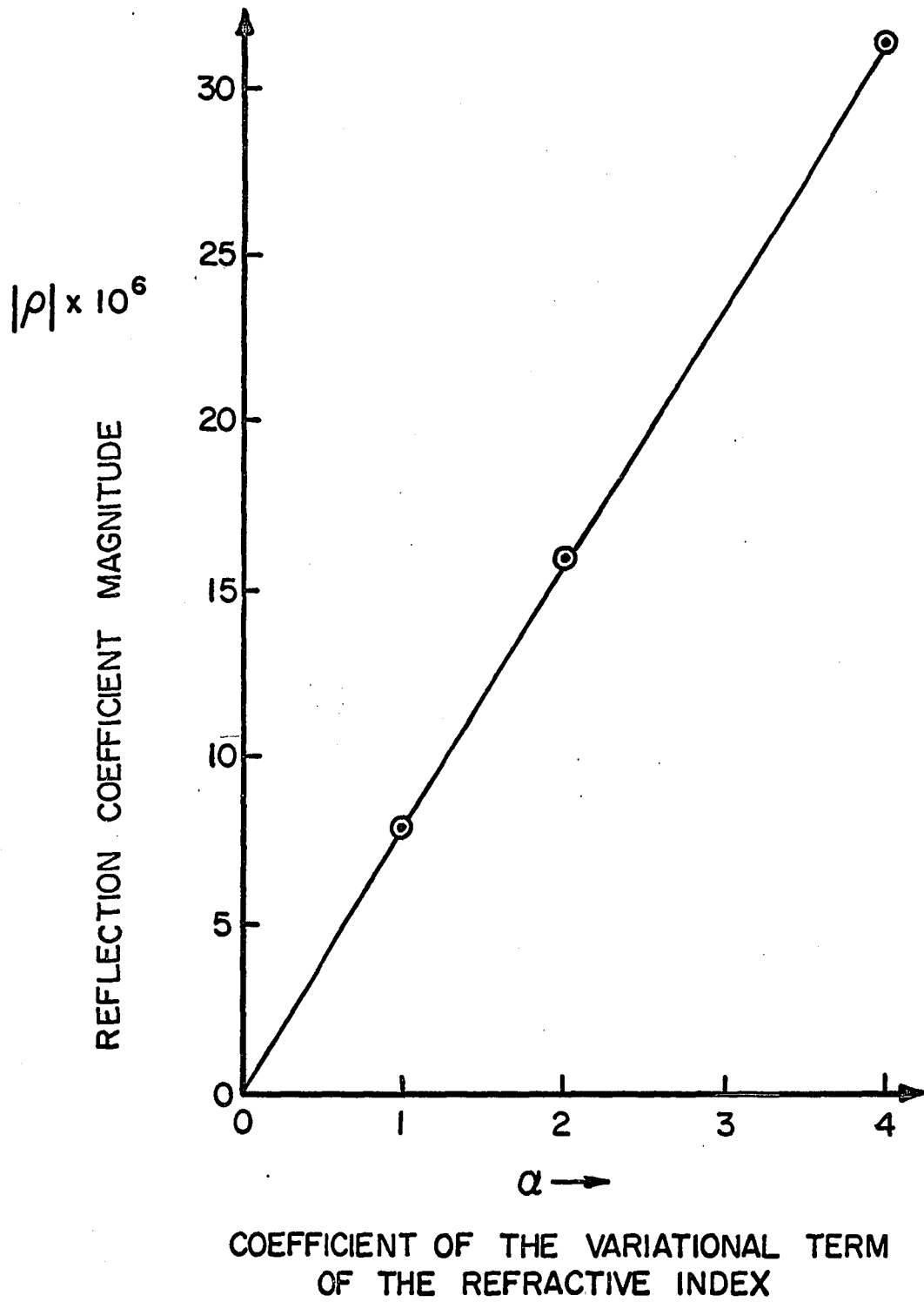 $\sin k_a x$.

COEFFICIENT OF THE VARIATIONAL TERM
OF THE REFRACTIVE INDEX

Figure 11. The graph of the magnitude of the
reflection coefficient to the number
of acoustic cycles or distance from
the origin.

REFLECTION COEFFICIENT MAGNITUDE $|\rho| \times (10^6)$
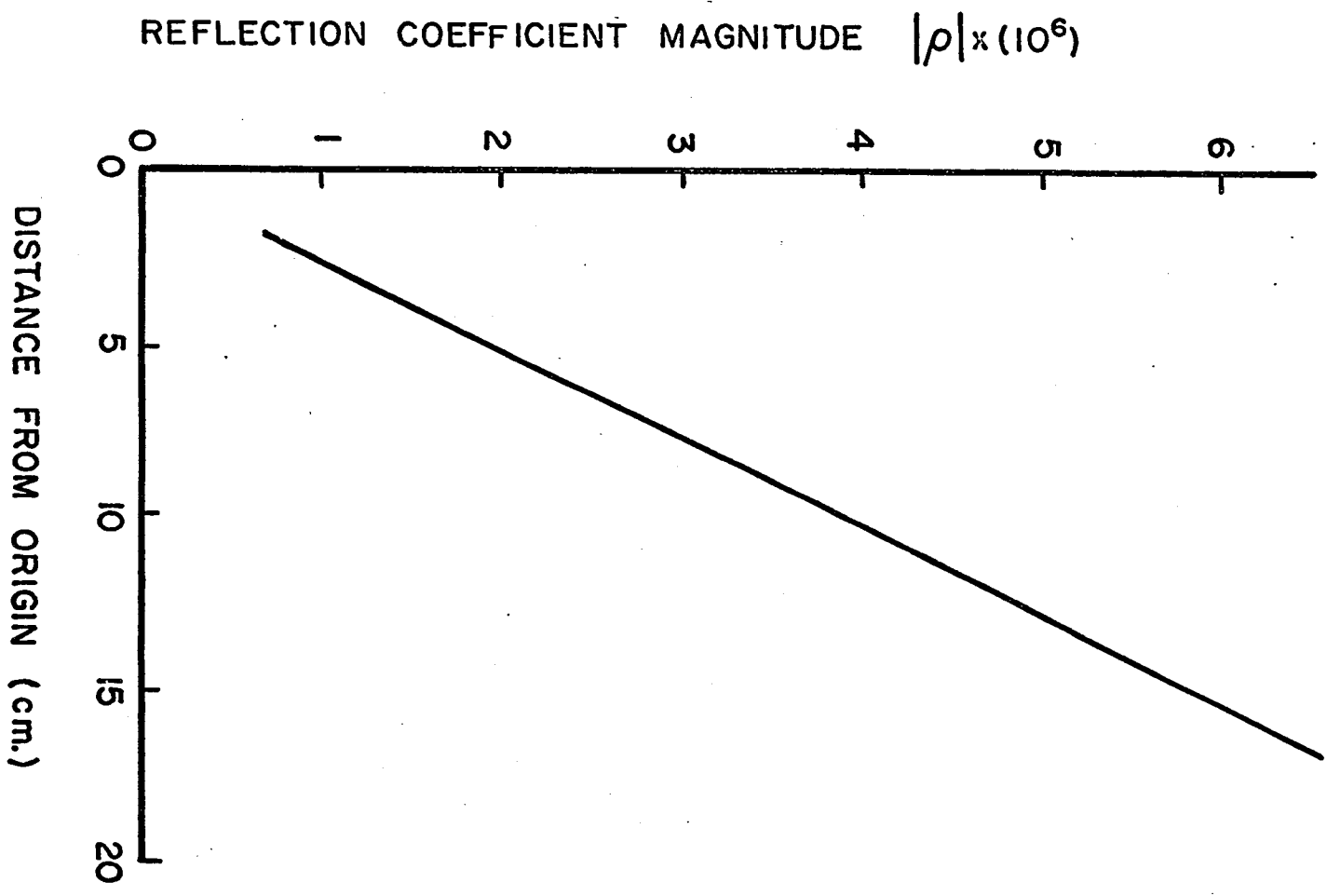

DISTANCE FROM ORIGIN (cm.)

Figure 12 shows a plot of the variation of the reflection coefficient versus the angle of incidence, that is, the angle that the electromagnetic wave makes as it impinges on the acoustic wave. The fractional change of the acoustic frequency is also indicated. It is apparent that no real change in the reflection coefficient occurs until the acoustic frequency changes about 3 percent; this change corresponds to an angle of incidence of about 12 degrees.

In the appendix, it is shown that the acoustic wave, as it propagates away from the source, gradually approaches the shape of a sawtooth wave with a sharp leading edge. For ease of numerical solution of the reflection coefficient from this wave shape, it was assumed that the function f could be described quite adequately in terms of the first three components of the Fourier series expansion of a sawtooth wave. The sum of the squares of the coefficients of the truncated Fourier series was set equal to the square of the magnitude of the sinusoidal acoustic wave used in the previous calculations, that is, a pressure wave which gave a refractive index of $n = 1 + 10^{-6} \sin k_a x$. By setting the "power" of the first three components equal to the total sinusoidal acoustic "power", the three Fourier components of f were evaluated.

To find the effect of the second and third harmonics on the reflection coefficient, the principle of superposition was imposed and the program run with the appropriate changes

Figure 12.   The graph of the reflection coefficient magnitude as a
function of the angle of incidence and the percent acoustic
frequency deviation.

REFLECTION
COEFFICIENT
MAGNITUDE

$|\rho| \times (10^6)$

ANGLE OF INCIDENCE (DEGREES)

PER CENT FREQUENCY DEVIATION
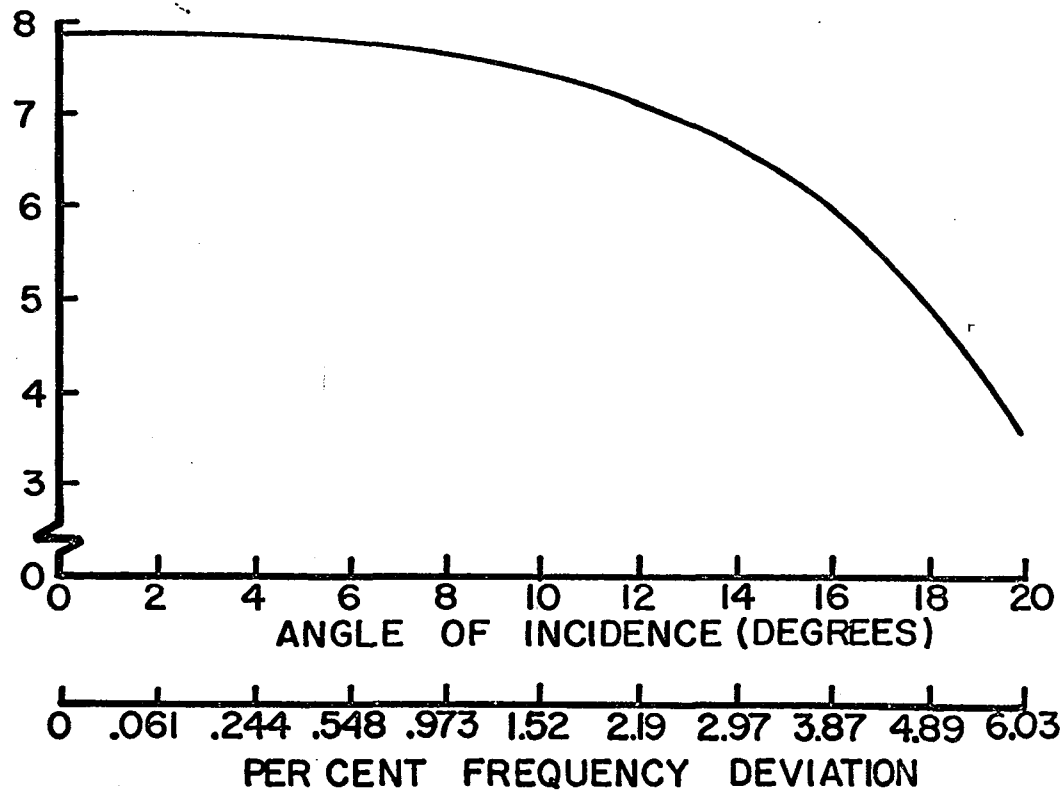
in the acoustic wave number (k_a). The resultant reflection coefficients are shown in Table 2.

Table 2. Comparison of reflection coefficient from a pure sinusoidal wave and a sawtooth acoustic wave

Refractive index variation	Reflection coefficient	db from Reference frequency
$2f(10^{-6})=10^{-6}\sin k_a x$	$7.86(10^{-6})$	0
$2f(10^{-6})=.855(10^{-6})\sin k_a x$	$6.73(10^{-6})$	-1.4
$2f(10^{-6})=.427(10^{-6})\sin 2k_a x$	$1.015(10^{-10})$	-97.8
$2f(10^{-6})=.285(10^{-6})\sin 3k_a x$	$1.11(10^{-10})$	-97.0

It is easily seen from these calculations that the higher order harmonics do not contribute significantly to the reflection coefficient. The assumption can be made that the high-powered acoustic wave, even though it propagates as a sawtooth-shaped pressure wave, can be considered to be a sinusoidal wave with an amplitude which is determined by the Fourier amplitude coefficient for the fundamental frequency of the wave.

Three examples of modulated acoustic waves will be considered next. The same interaction length as used for the sinusoidal wave will be used to check the procedure and formulation in the first two cases, but the last example will consider a longer interaction length.

The first wave form to be considered is that of a modulated wave where $2f(10^{-6}) = 10^{-6} \sin 5\pi x \sin 200\pi x$. This wave form is shown in Figure 6. The real part of the perturbed field E_1 is shown as Figure 13. The magnitude of the reflection coefficient of such a wave is $|\rho| = 5.0(10^{-6})$, a smaller magnitude than that of the sinusoidal wave.

The second example is that of a damped sinusoidal wave of the form $2f(10^{-6}) = 10^{-6}e^{-20x}\sin 200\pi x$, as illustrated in Figure 7. The damping factor fo 20 was chosen since it approximates the decay of an acoustic burst following shutdown of a mechanical acoustic source of the type used by Fetter, Smith, and Klein (6). A graph of the real part of the perturbed field is given in Figure 14. The reflection coefficient was found to be $1.88(10^{-6})$ for L = .2 meters.

The last example is that of the modulated wave illustrated in Figure 8. The wave-is a constant magnitude sinusoidal wave of the form $2f(10^{-6}) = 10^{-6}\sin 200\pi x$ for a length, L = 0.2 meters, from the origin, followed by a damped sinusoidal wave given as $2f(10^{-6}) = 10^{-6}e^{-20(x-.2)}\sin 200\pi(x-.2)$ to a distance of 0.4 meters. The increased interaction length of the wave shape over the sinusoidal wave leads to an increase in the magnitude of the reflection coefficient to $9.75(10^{-6})$.

Because the interaction length (L) is chosen to include an integral number of acoustic wave lengths, the reflection coefficient has only a very small phase angle associated with

Figure 13. The graph of the variation of the real part of the perturbed field caused by a modulated acoustic wave of the form $10^{-6} \sin 5\pi x \sin 200\pi x$.
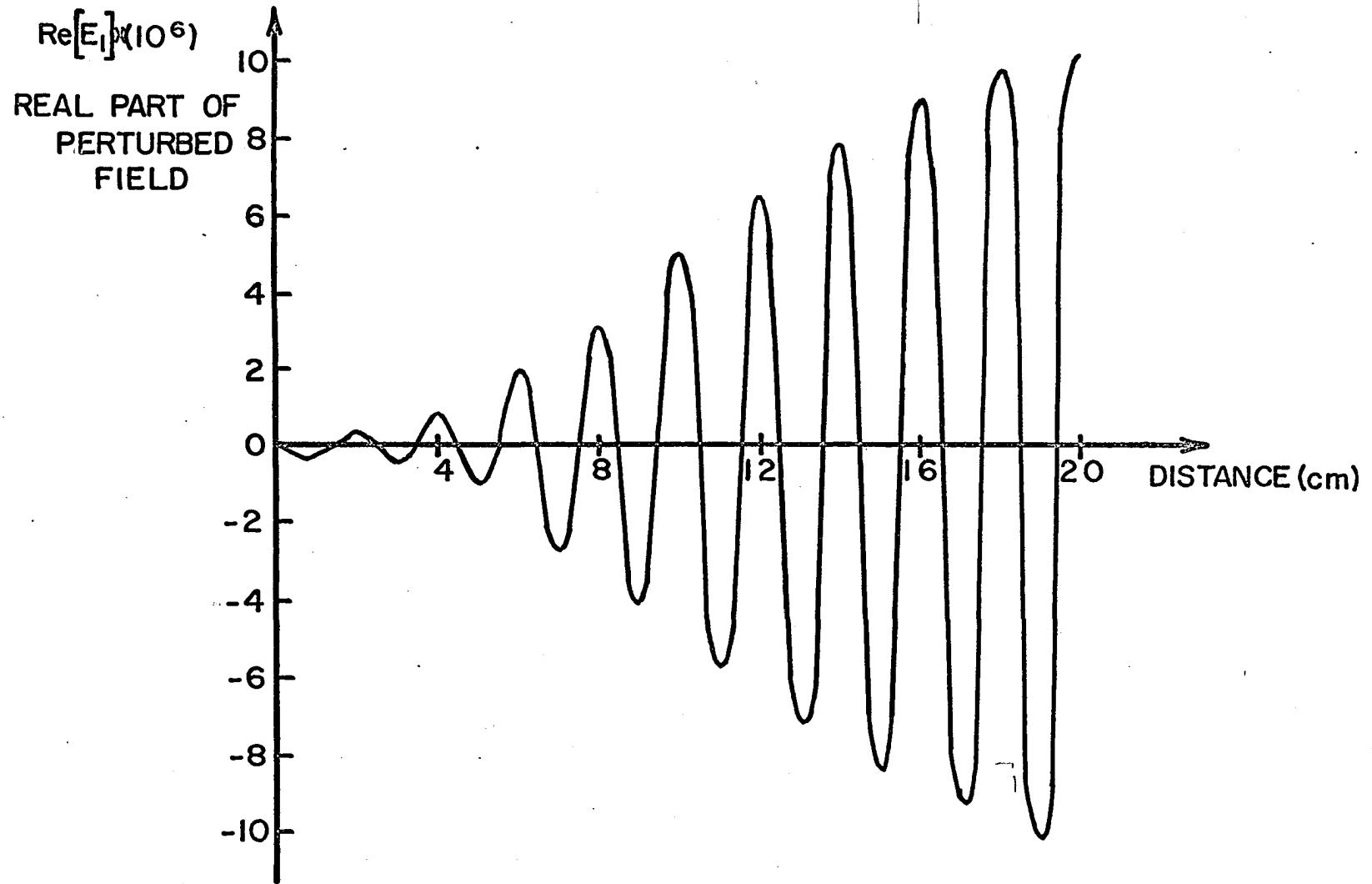
Figure 14. The graph of the variation of the real part of the
perturbed field caused by a modulated acoustic wave
of the form $10^{-6}e^{-20x}\sin 200\pi x$.
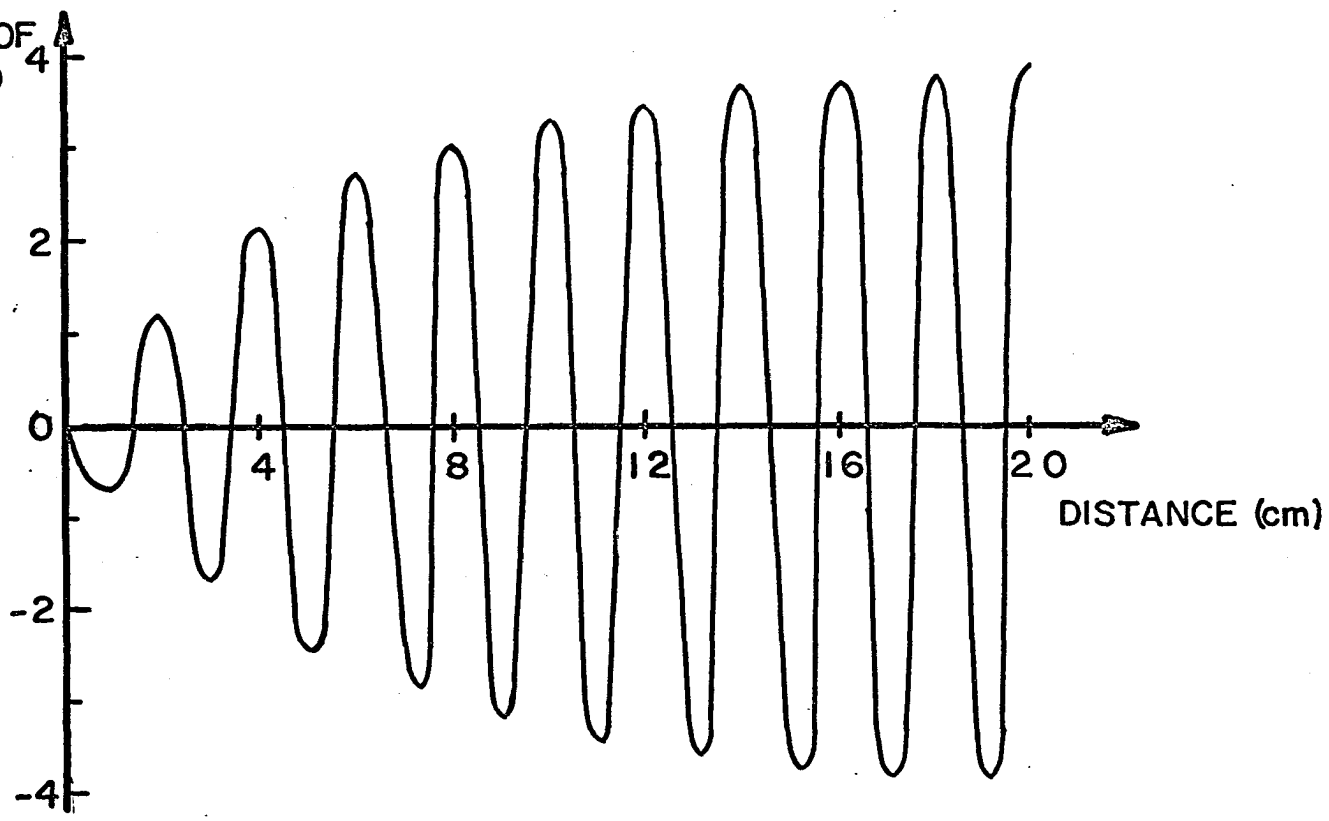
it. The graphs of the perturbed fields show that, if the acoustic wave were terminated at any point other than at an integral number of acoustic wave lengths, the reflection coefficient would have a phase angle which is determined by the real and imaginary parts of E_1 at the point of termination.

CONCLUSIONS

A general formulation of a mathematical model to describe
the interaction of an electromagnetic wave and a known one-
dimensional distribution of the refractive index is presented.
The model is of a form which is easily solved by the use of
numerical techniques.

A sinusoidal pressure distribution was assumed in order to
check the model and the boundary conditions with published
results of other models. Agreement is found between the pub-
lished and computed results.

The refractive index was approximated by a stair-step
function to obtain an estimate of the incremental reflection.
In the limit of differential step size, an estimate of the total
reflection was determined by integration over various inter-
action lengths. This analytical technique indicated that the
reflection coefficient varied linearly with both the pressure
and the interaction length. Quantitative values of the reflec-
tion coefficient were not obtained readily by this approach.
As predicted, the numerically computed reflection coefficient
was found to vary in a linear manner with the magnitude of
pressure as well as with the interaction length.

The reflection coefficient varies with the frequency
of the acoustic wave as well as with the angle of incidence
that the electromagnetic wave makes with the acoustic wave.

The frequency variations of the acoustic signal and the angle of incidence have the same apparent effect on the reflection coefficient in the limit of small angles of incidence. Variations of the reflection coefficient with frequency and angle of incidence were calculated for a sinusoidal pressure distribution. It was found that the angle of incidence must be on the order of 10 to 12 degrees before the reflection coefficient varied appreciably. This is equivalent to a frequency deviation of the acoustic wave of two percent or less from the frequency of maximum reflection.

Mechanical methods are, conventionally, used to measure parameters of the atmosphere. Mechanical instruments have inherent disadvantages, such as finite response time, expense of construction, and the necessity of supporting structures. It is of great interest to find some method of measuring remotely the instantaneous parameters of the atmosphere.

The propagation characteristics of an acoustic wave launched into the atmosphere depend on the atmospheric parameters. If the propagation of the acoustic wave could be monitored, the parameters, hopefully, could be measured. One logical method of remotely monitoring the behavior of the acoustic wave involves the study of radar reflection from the atmospheric disturbance caused by the acoustic wave in a combination acoustic-electromagnetic probe arrangement. Before such a probe would be useful, however, it is necessary

to know how the electromagnetic wave interacts with the acoustic wave and which parameters of the acoustic wave have the greatest effect on the electromagnetic wave.

Many studies of the reflection of electromagnetic energy from a known spatial variation of the dielectric constant have been made. For the most part, investigators have considered only those problems where the relative permittivity varies quite radically from a constant. However, variations of the refractive index in the atmosphere are, in general, very small. To simplify the solution of the problem of a small variation of the refractive index, a perturbation technique was found which is easily adapted to standard methods of numerical solution. The formulation of the general problem is such that if the acoustic wave shape or any other spatial distribution of the refractive index is known, the reflection coefficient can be found. Acoustic power levels necessary to obtain a refractive index variation in the order of 10^{-6} involve non-linearities in the atmosphere with the effect of reducing the reflected electromagnetic energy to a value well below that expected from this type of wave in the linear atmosphere. The approach developed in this thesis permits the determination of the reflection from any arbitrarily-shaped, spatial index of refraction.

Several extensions of this work should be carried out before the results of acoustic-electromagnetic probe

measurements will be meaningful. These extensions include the effects of atmospheric non-linearities and turbulence on the acoustic wave and the question of experimental methods which could be used to study the effect of the atmosphere on the acoustic wave.

LITERATURE CITED

1. Barrar, R. B. and Redheffer, R. M. On nonuniform dielectric media. Institute of Radio Engineers Transactions on Antennas and Propagation AP-3: 101-107. 1955.

2. Blackstock, D. T. Propagation of plane sound waves of finite amplitude in nondissipative fluids. Acoustical Society of America Journal 34: 9-30. 1962.

3. Brillouin, L. Wave propagation in periodic structures. 2nd edition. New York, New York, Dover Publications. c1953.

4. Courant, R. and Friedrichs, K. O. Supersonic flow and shock waves. New York, New York, Interscience Publications. 1948.

5. Fay, R. D. Plane sound waves of finite amplitude. Acoustical Society of America Journal 3: 222-241. 1931.

6. Fetter, R. W., Smith, P. L., and Klein, V. Design concept study for a wind profile instrument. United States Army Rocket and Guided Missile Agency Report No. DA-23-072-506-ORD-11. 1961.

7. Jeffrey, A. and Taniuti, T. Non-linear wave propagation. In Richard Bellman, ed. Mathematics in science and engineering. Volume 9. New York, New York, Academic Press. 1964.

8. Jones, B. L. and Patton, P. C. Solution of reflection problems by means of a transmission line analogy. Institute of Radio Engineers Transactions on Antennas and Propagation AP-8: 418-422. 1960.

9. Liepmann, H. W. and Roshko, A. Elements of gas dynamics. New York, New York, John Wiley and Sons. 1957.

10. Ramo, S. and Whinnery, J. R. Fields and waves in modern radio. 2nd edition. New York, New York, John Wiley and Sons. 1960.

11. Richmond, J. H. Transmission through inhomogeneous plane layers. Institute of Radio Engineers Transactions on Antennas and Propagation AP-10: 300-305. 1962.

12. Schmitt, H. J. and Wu, T. T. Electromagnetic reflection from sound waves. I. Acoustical Society of America Journal 32: 1660-1667. 1960.

13. Schmitt, H. J. and Wu, T. T. Electromagnetic reflection from sound waves. II. Acoustical Society of America Journal 36: 437-445. 1964.

14. Smith, E. K., Jr. and Weintraub, S. The constants in the equation for atmospheric refractive index at radio frequencies. Institute of Radio Engineers Proceedings 41: 1035-1037. 1953.

15. Smith, P. L. and Rogers, R. R. On the possibility of radar detection of clear-air turbulence. Unpublished mimeographed paper. Kansas City, Missouri, Midwest Research Institute. ca. 1964.

16. Webster, A. G. Partial differential equations of mathematical physics. 2nd edition. New York, New York, Dover Publications. 1955.

ACKNOWLEDGMENTS

APPENDIX A

This appendix, for purposes of completeness, will describe the media in which the acoustic wave must propagate. The atmosphere, for the cases of the type considered in this thesis, can be described by the fluid continuity and momentum equations and by its thermodynamic properties. The thermodynamic properties of the atmosphere are contained in the expression for the velocity of sound and will not be considered explicitly.

To analyze acoustic wave propagation, small signal acoustics will be considered first; an extension then will be made to higher-powered acoustic waves. For simplicity, only the one-dimensional problem will be considered.
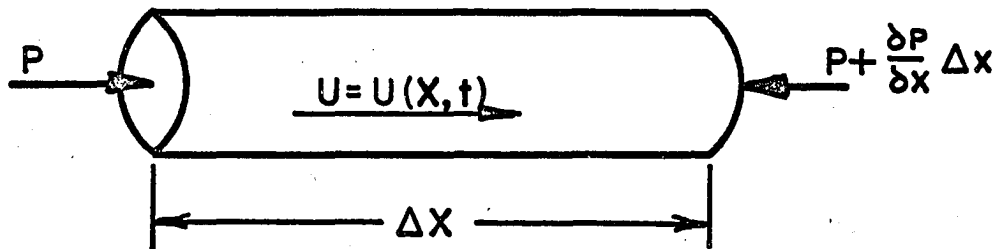
Derivation of Continuity Equation

Continuity in a fluid dynamic system means that mass can be neither created nor destroyed. It follows that the difference between the mass flow into and out of a unit volume per unit time must be equal to the time rate of change of the density within the volume.

When a cylindrical section such as is shown in Figure 15, having a unity cross-sectional area and a length Δx, is considered, it is easily seen that the mass flow into the section per unit time is ρu, where ρ is the mass density, and u is the x direction velocity. Similarly, the mass flow out of the cylindrical section per unit time is $(\rho + \frac{\partial \rho}{\partial x} \Delta x)$

Figure 15. Schematic representation of the volume
used for the derivation of the one-
dimensional continuity equation.

Figure 16. Schematic representation of the volume
used for the derivation of the one-
dimensional momentum equation.

$$U \qquad\qquad\qquad\qquad U + \frac{\partial U}{\partial x}\Delta X$$

$$\rho \qquad\qquad\qquad\qquad \rho + \frac{\partial \rho}{\partial x}\Delta X$$

$$\longleftarrow \Delta X \longrightarrow$$

$$P \qquad U = U(X, t) \qquad P + \frac{\partial P}{\partial x}\Delta X$$

$$\longleftarrow \Delta X \longrightarrow$$
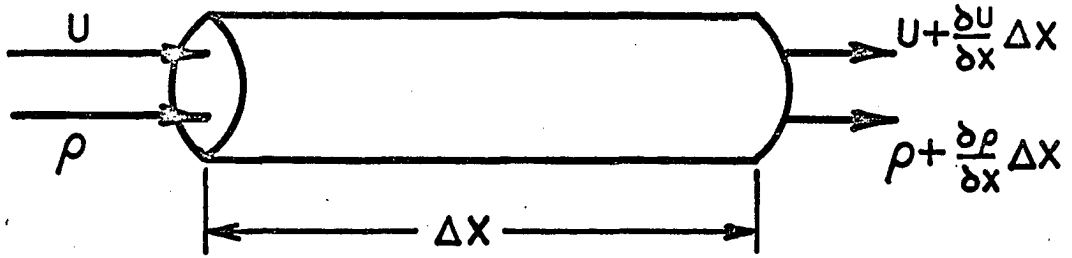
$(u + \frac{\partial u}{\partial x} \Delta x)$. The condition of continuity implies that

$\rho u - [u + \frac{\partial u}{\partial x} \Delta x][\rho + \frac{\partial \rho}{\partial x} \Delta x] = \frac{\partial \rho}{\partial t} \Delta x$. If second-order product

terms are neglected, this equation reduces to the one0dimension-

al form of the continuity equation:

$$\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = - \frac{\partial \rho}{\partial t} \qquad (1\text{-}A)$$

Derivation of Momentum Equation

The force acting on a fixed mass of fluid is a conse-

quence of a pressure differential across the volume of fluid.

This force gives the mass an acceleration in the direction

of the force. Since the velocity of a compressible fluid

can be a function of both position and time, the total

acceleration of a mass $\rho \Delta x$ in a one-dimensional system, such

as shown in Figure 16, is:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$$

The net force acting on a mass $\rho \Delta x$ is: $[P - (P + \frac{\partial P}{\partial x} \Delta x)]$

$= - \frac{\partial P}{\partial x} \Delta x$. Newton's Law equates these two forces to give:

$$\rho \Delta x (u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}) = - \frac{\partial P}{\partial x} \Delta x$$

Division of this equation by $\rho \Delta x$ gives the one-dimensional

momentum equation:

$$u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial P}{\partial x} \qquad (2\text{-}A)$$

Derivation of the Equation for the Speed of Sound

The speed of propagation of a small disturbance in a gas is, by definition, the speed of sound (C_a). The disturbance will be represented in terms of its velocity, its density, and its pressure. These quantities are interrelated and spatially dependent as indicated in Figure 17.

In a steady flow, that is, a fluid flow which has a constant velocity and density, the continuity Equation 1-A can be written as $- V \frac{\partial \rho}{\partial x} = \rho \frac{\partial V}{\partial x}$, where V is the velocity of the disturbance. The change in momentum then can be expressed as:

$$V^2 \rho + P = \left(V + \frac{\partial V}{\partial x} \Delta x\right)^2 \left(\rho + \frac{\partial \rho}{\partial x} \Delta x\right)$$

Neglection of the second-order terms reduces these equations to:

$$2V\rho \frac{\partial V}{\partial x} + V^2 \frac{\partial \rho}{\partial x} + \frac{\partial P}{\partial x} = 0$$

The continuity equation just given, used in conjunction with the momentum equation, gives the square of the velocity of propagation of the disturbance:

$$V^2 = \frac{\partial P}{\partial \rho} \tag{3-A}$$

If the process is adiabatic, then $C_a = V = \frac{\gamma P}{\rho}$, where, for the atmosphere, $\gamma = 1.40$.

Small Signal Acoustics

The momentum and continuity equations plus the velocity of sound describe the media in which the acoustic wave propagates. These equations are sufficient to allow a solution

Figure 17. Schematic representation of a small
discontinuity used for the derivation of
the acoustic speed of sound.

$$V + \frac{\partial V}{\partial x}\Delta x$$

$$\rho + \frac{\partial \rho}{\partial x}\Delta x$$

$$P + \frac{\partial P}{\partial x}\Delta x$$

$$V$$

$$\rho$$

$$P$$
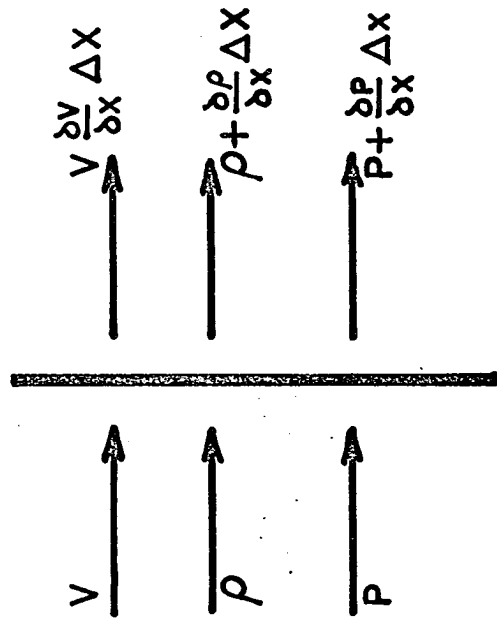
for particle velocity (u), the density (ρ), and the pressure
(p).

To begin a study of finite amplitude pressure wave propagation, it is first necessary to make a study of small signal acoustic waves. From a consideration of small signal acoustics, considerable information can be found which will be useful in the derivation of the propagation characteristics of finite amplitude pressure waves.

The difference between small signal and finite amplitude pressure waves is that, for small amplitude waves, the variations of u, ρ, and p are small, and the non-linear terms in the momentum and continuity equations can be neglected; for finite amplitude waves, these small signal approximations are not valid.

The condensation s is defined as:

$$s = \frac{\rho - \rho_o}{\rho_o} \tag{4-A}$$

where ρ_o is the static density and ρ is the static density plus the instantaneous variation in the density due to the acoustic wave. By definition, signal waves require that s $\ll$ 1. The speed of sound is also considered to be constant for the small signal case.

The pressure term in the momentum equation can be rewritten by making use of the equation for the speed of sound, as follows: $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial x} = C^2 \frac{\partial P}{\partial x}$. The momentum Equation 2-A and continuity Equation 1-A can be written as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial x} = 0 \qquad (5\text{-}A)$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \qquad . \qquad (6\text{-}A)$$

Substitution of ρ from Equation 4-A into Equations 5-A and 6-A reduces the momentum and continuity equations to:

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} + s \frac{\partial u}{\partial x} + u \frac{\partial s}{\partial x} = 0 \qquad (7\text{-}A)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{c_o^2}{1+s} \frac{\partial s}{\partial x} = 0 \qquad (8\text{-}A)$$

Using the small signal approximations that $s \ll 1$ and that the speed of sound is constant at a value of C_o and neglecting the non-linear terms, $s \frac{\partial u}{\partial x}$, $u \frac{\partial u}{\partial x}$, and $u \frac{\partial s}{\partial x}$, simplify Equations 7-A and 8-A to:

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad (9\text{-}A)$$

$$\frac{\partial u}{\partial t} + c_o^2 \frac{\partial s}{\partial x} = 0 \qquad (10\text{-}A)$$

Equations 9-A and 10-A can be further reduced to the classical acoustic wave equation in terms of either u or s:

$$\frac{\partial^2 s}{\partial t^2} - c_o^2 \frac{\partial^2 s}{\partial x^2} = 0 \qquad (11\text{-}A)$$

$$\frac{\partial^2 u}{\partial t^2} - c_o^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (12\text{-}A)$$

Solution of Equations 11-A and 12-A indicates that small amplitude acoustic signals should propagate as unattenuated and undistorted waves. Forward-traveling waves of the form $u = f(C_o t\text{-}x)$ and $s = g(C_o t\text{-}x)$ are possible solutions of Equations 11-A and 12-A. Using these solutions in Equation

9-A gives the relationship between s and u as:

$$C_o s = u \qquad (13\text{-}A)$$

Use of the adiabatic condition shows that:

$$\frac{P_o}{P_o} (1+s)^\gamma \cong (1+\gamma s) \quad \text{or} \quad \gamma s = \frac{\Delta P}{P_o} \qquad (14\text{-}A)$$

Substituting Equation 13-A into 14-A gives a relationship for the particle velocity as a function of pressure:

$$\Delta P = u\, \rho_o C_o \qquad (15\text{-}A)$$

Since ρ_o and C_o are constants, it follows from Equation 12-A that the pressure change ΔP must also satisfy the same wave equation as u and s.

The acoustic intensity can now be determined. By definition, the small signal acoustic intensity in watts/cm^2 is the time average of the product of the acoustic particle velocity and the acoustic pressure. If the pressure and velocity waves have no phase difference, the acoustic intensity is expressed as:

$$I = \frac{(\Delta P)^2}{2C_o \rho_o} \qquad (16\text{-}A)$$

where ΔP is the magnitude of the sinusoidal pressure wave, and I is the intensity of the acoustic wave.

The reason for considering the small amplitude analysis is to find whether or not a "small acoustic signal" will give an adequate variation of the refractive index to be used in an acoustic electromagnetic probe.

The empirical formula for the refractive index is expressed as $n = 1 + \frac{79P}{T}10^{-9}$, where T is the temperature in degrees Kelvin, and P is the pressure in dynes/cm^2. Typical expressions of n are assumed in this thesis to be of the form $n = 1 + 10^{-6}Q(x)$, where Q(x) is a real function of position with a magnitude of the order of unity. The coefficient 10^{-6} of the spatial variation of n corresponds to a pressure variation ΔP of $3.71(10^3)$ dynes/cm^2. This pressure corresponds to an intensity level of approximately 142 db. The reference level for measuring acoustic intensity is taken as 10^{-16} watts/cm^2, the threshold of audibility of an average individual. In comparison, the average conversation intensity level is 30 db. The corresponding particle velocity is 9.905 meters/ second. From these calculations, it can be seen that neglecting particle velocity in the momentum and continuity equations is subject to serious question. A more accurate analysis of the acoustic wave can be made by using the method of characteristics.

Finite Amplitude Acoustic Waves

The method of characteristics gives a solution of the momentum and continuity Equations, 5-A and 6-A, without neglecting the non-linear terms.

A complete solution will not be presented here, inasmuch as Blackstock (2) has given a detailed discussion of this type of rather involved problem. It will be shown that the acoustic

wave does not propagate as an undistorted wave; and, as it propagates away from the generator, it assumes a sawtooth shape.

Essentially, the method of characteristics is a mathematical method that can be used to solve equations of the form $f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} = h$, where f, g, and h can be functions of u, x, and y. Detailed discussions of this type of mathematical analysis are given by Courant (4), Jeffrey (7), Liepmann (9), and Webster (16).

An extremely simplified explanation of this method will be described. To start the solution, the assumption is made that $w_{(x,y,z)} = u_{(xy)} - z$ is known and that this function describes a continuous surface in x, y, and z space as illustrated in Figure 18. The ratio of the components of the gradient of this surface u at some point x, y, z is $\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} : -1$. In vector notation, the unit normal $\hat{a}_n$ can be written as $\hat{a}_n = \hat{\imath} \frac{\partial u}{\partial x} + \hat{\jmath} \frac{\partial u}{\partial y} - \hat{k}$. If a line tangent to the surface at x, y, z is described by the vector $\bar{a}_t = \hat{\imath} \frac{\partial u}{\partial x} + \hat{\jmath} \frac{\partial u}{\partial y} - \hat{k}$, then $\hat{a}_n \cdot \bar{a}_k = f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} - h = 0$, which is the equation to be solved.

If ds is a segment of the tangent curve C traced by $\bar{a}_k$, the projection of ds upon any axis is the product of ds and the direction cosine of the angle between ds and the axis of interest. Thus, for the point $w = u_{(xy)} - z = 0$, $\frac{dx}{ds} = f$, $\frac{dy}{ds} = g$ and $\frac{dz}{ds} = h$. If just the xy plane is considered,

Figure 18. Diagram of continuous surface
$w = u(xy) - z$.
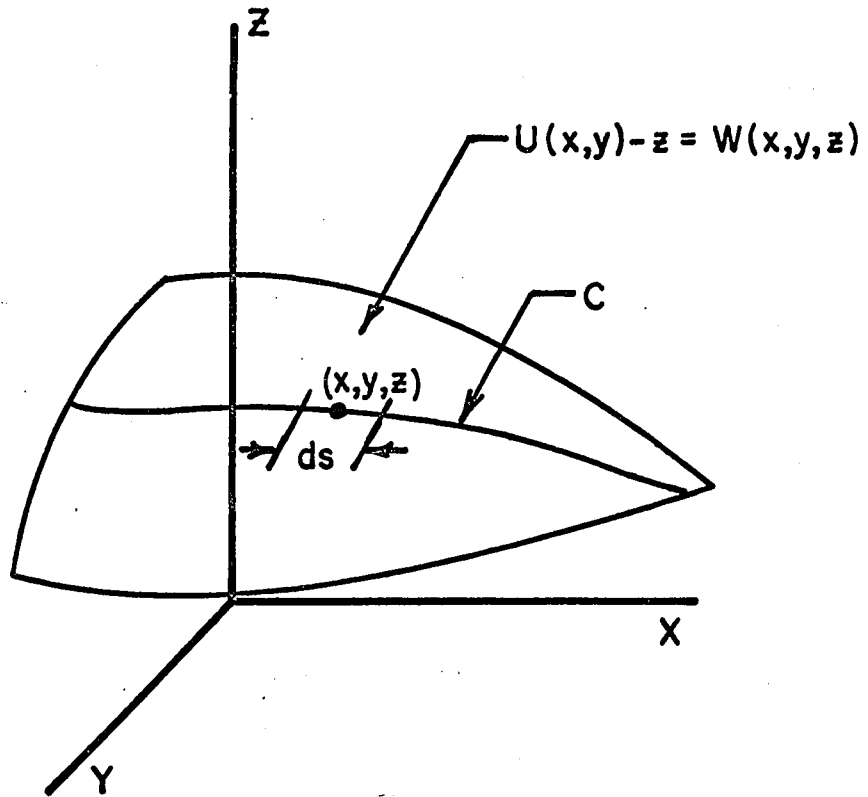
$$\frac{dx}{ds}\Big|_{\frac{dy}{ds}} = f/g = \frac{dx}{dy} \qquad\qquad (17\text{-}A)$$

The continuity and momentum Equations, 5-A and 6-A, can be reduced to the form $f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} - h = 0$ by substitution of an expression for the particle velocity which can be obtained readily from an extension of small signal acoustics.

Figure 19 illustrates a wavelet superimposed upon a small signal acoustic wave. The wavelet will not travel at the same speed as the small signal wave. For example, at x_1 the density of the wave is higher than the average; then $C_1 = C_o (\frac{\rho_1}{\rho_o})^{\frac{\gamma-1}{2}}$ is larger than C_o. Thus, when the density of the small amplitude wave increases, the speed of the wavelet increases. Conversely, as the density decreases, the wavelet speed tends to slow down.

The particle velocity of the wavelet from Equation 13-A is $C\, s_1 = u_1$, where u_1 is the change in the particle velocity from that of the small signal wave. If the wavelet is considered, Equation 13-A can be written as $C \frac{\Delta\rho}{\rho} = \Delta u$ as an approximation. In the limit, u becomes

$$u = \int_{\rho_o}^{\rho} C \frac{d\rho}{\rho} \qquad\qquad (18\text{-}A)$$

Equation 18-A, because of the limits, gives the total velocity of the particles. If the process is adiabatic, Equation 18-A can be integrated; and u is found to be:

$$u = \frac{2}{\gamma-1} (C-C_o) \qquad\qquad (19\text{-}A)$$

Figure 19.   Schematic representation of a wavelet
             superimposed on a sinusoidal small
             amplitude acoustic wave.
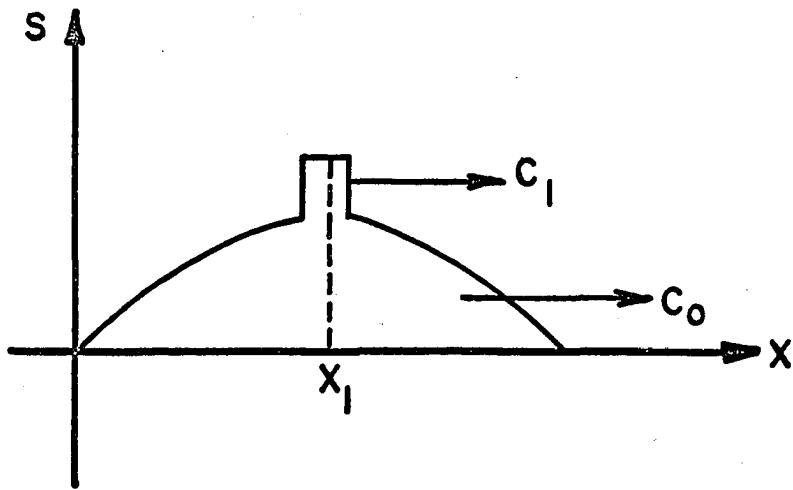
Combining Equation 19-A and the equation for the velocity of sound gives an expression for pressure of:

$$P = P_o \left(\frac{C_o + \frac{\gamma-1}{2} u}{C_o} \right)^{2\gamma/\gamma-1} \tag{20-A}$$

Similarly, by the use of the equation for the velocity of sound and an adiabatic process, Equation 19-A can be solved for the density ρ. Substituting the density ρ for $\frac{\partial \rho}{\partial x}$ and $\frac{\partial \rho}{\partial t}$ in Equations 5-A and 6-A reduces these two equations to a single equation as a function of u and C:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + C \frac{\partial u}{\partial x} = 0 \tag{21-A}$$

This equation can be solved by the method of characteristics, that is, f, g, and h are easily identified as f = u + C, g = 1, and h = 0, respectively. Therefore, Equation 17-A is:

$$\frac{dx}{dt} = u \pm C \tag{22-A}$$

where the plus sign corresponds to waves traveling in the positive x-direction and the negative sign to waves traveling in the negative x-direction. Substitution of C as a function of C_o and u from Equation 19-A reduces Equation 22-A to:

$$\frac{dx}{dt}\bigg|_{\text{for fixed } u} = C_o + \tfrac{1}{2} (\gamma+1)u \tag{23-A}$$

for positive-traveling waves. For a given u distribution along one of the axes of the x,t plane, the basic wave shape can be found. A diagram of the process that might be used to find the u distribution is shown as Figure 20. When the slopes of the constant velocity (u) cross, the solution

Figure 20.   Schematic representation of the progressive
             distortion of a finite amplitude acoustic
             wave as it propagates away from the generator.

DISTANCE FROM ACOUSTIC SOURCE

$\frac{dx}{dt} = C_0 - \frac{\gamma+1}{2} U$

$\frac{dx}{dt} = C_0$

$\frac{dx}{dt} = C_0 + \frac{\gamma+1}{2} U$

$X = K \cos \omega t$

$U = -K \sin \omega t$

78

ceases to have physical meaning because it is assumed that the velocity cannot be double valued. The velocity wave might look like the wave shown in the top part of Figure 20.

The pressure wave will be similar to the velocity wave. This fact is evident if the pressure Equation 20-A is expanded by the binomial expansion $P = P_o (1 + \frac{\gamma u}{C_o})$. Thus, in the limit of large distances and long propagating times, a finite amplitude sinusoidal acoustic wave degenerates into a sawtooth wave.

Fourier Series Approximation of the
Finite Amplitude Acoustic Wave

It was shown previously that for finite amplitude acoustic waves, the pressure wave form degenerates to a sawtooth-shaped wave. To analyze the effect this wave form has upon the reflection coefficient, a Fourier analysis was made of the pressure wave.

The coefficients of the Fourier series were evaluated by equating the "power" in the assumed sinusoidal wave to the "total power" in the coefficients of the Fourier series. Equating the power in the series expansion is equivalent to equating the sum of the squares of the pressures of each of the harmonics of the series to the pressure squared of the sinusoidal wave.

The Fourier series for the finite amplitude wave is

$f(x) = \sum_1^\infty (-1)^n \frac{1}{n} A_n \sin n\, k_a x$. If, as an approximation to

the pressure wave, the first three harmonics are used, then the following function approximates the sawtooth wave form:

$f(x) = -A k_a x + \frac{A}{2} \sin 2 k_a x - \frac{A}{3} \sin 3 k_a x$. If P_T is the pressure of the assumed sinusoidal acoustic wave which gives a refractive index variation of $n = 1 + 10^{-6} \sin k_a x$, the first coefficient is $A = \frac{6}{7} P_T$. The Fourier series for the calculated pressure which corresponds to the 10^{-6} coefficient of the refractive index is:

$$f(10^{-6}) = 10^{-6}[-.855 \sin k_a x + .427 \sin 2 k_a x$$
$$- .285 \sin 3 k_a x] \quad (24\text{-}A)$$